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Strongly Gorenstein projective, injective, and flat modules

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Abstract

In this paper, we study a particular case of Gorenstein projective, injective, and flat modules, which we call, respectively, strongly Gorenstein projective, injective, and flat modules. These last three classes of modules give us a new characterization of the first modules, and confirm that there is an analogy between the notion of "Gorenstein projective, injective, and flat modules" and the notion of the usual "projective, injective, and flat modules".

MSC: 13C14

1. Introduction

Throughout this work, R is a commutative ring with identity element, and all R-modules are unital. If M is any R-module, we use $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the usual projective and flat dimensions of M, respectively. It is convenient to use "local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal.

In 1967–69, Auslander and Bridger [2,3] introduced the G-dimension for finitely generated modules over Noetherian rings, denoted by G-dim(M) where M is a finitely generated module. They proved the inequality G-dim$(M) \leq \mathrm{pd}(M)$, with equality G-dim$(M) = \mathrm{pd}(M)$ when $\mathrm{pd}(M)$ is finite. We say that G-dimension is a refinement of projective dimension.
Several decades later, Enochs, Jenda, and Torrecillas [10–12] extended the ideas of Auslander and Bridger, and introduced three homological dimensions, called Gorenstein projective, injective, and flat dimensions, which have all been studied extensively by their founders and by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others [4,8,13,15,17]. They proved that these dimensions are similar to (and refinements of) the classical homological dimensions; i.e., projective, injective, and flat dimensions, respectively.

The Gorenstein projective, injective and flat dimension of a module is defined in terms of resolutions by Gorenstein projective, injective and flat modules, respectively.

Definition 1.1 (*[17]*).

(1) An R-module M is said to be Gorenstein projective (G-projective for short), if there exists an exact sequence of projective modules

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$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.

The exact sequence $\mathbf{P}$ is called a complete projective resolution.

(2) The Gorenstein injective (G-injective for short) modules are defined dually.

(3) An R-module M is said to be Gorenstein flat (G-flat for short), if there exists an exact sequence of flat modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $- \otimes I$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective module. The exact sequence $\mathbf{F}$ is called a complete flat resolution.

Almost by definition one has the inclusion

$$\{\text{projective modules}\} \subseteq \{G\text{-projective modules}\}.$$

The main idea of this paper is to introduce and study an intermediate class of modules called strongly Gorenstein projective modules (SG-projective for short),

$$\begin{aligned}\{\text{projective modules}\} &\subseteq \{\text{SG-projective modules}\}\\ &\subseteq \{G\text{-projective modules}\}.\end{aligned}$$

These modules are defined by considering the situation where all modules and homomorphisms of the complete resolutions of Definition 1.1(1) above are equal (see Definition 2.1). Similarly, we define the strongly Gorenstein injective, and flat modules (SG-injective, and SG-flat, respectively, for short) (see Definitions 2.1 and 3.1).

The simplicity of these modules manifests in the fact that they are simpler characterizations than their corresponding Gorenstein modules (see Propositions 2.9, 2.12 and 3.6 and Remark 2.10(2)). Moreover, with such modules, we are able to give nice new characterizations of Gorenstein projective, injective, and flat modules, similar to the characterization of projective modules by the free modules, which is the main result of this paper (see Theorems 2.7 and 3.5):

Theorem. *A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*
Every flat module is a direct summand of a strongly Gorenstein flat module.

Over Noetherian rings the Gorenstein projective, injective, and flat modules were (and still are) excessively studied (please see [7]). So, we find that the relation that exists between the Gorenstein projective and Gorenstein flat modules is (nearly) similar to the one between the classical projective and flat modules (see [7, Proposition 5.1.4][1] and [7, Theorem 5.1.11]). In [17], Holm extended [7, Proposition 5.1.4] to coherent rings with finite finitistic projective dimension. Recall the finitistic projective dimension of a ring R, $\mathrm{FPD}(R)$, is defined by

$$\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R(M) \mid M\ R\text{-module with } \mathrm{pd}_R(M) < \infty\}.$$

Proposition 1.2 (*[17], Proposition 3.4*). *If R is coherent with finite finitistic projective dimension, then every Gorenstein projective R-module is Gorenstein flat.*

Also, [7, Theorem 5.1.11] can be extended to coherent rings. In fact, using Holm's work [17], the same proof of [7, Theorem 5.1.11] and [7, Lemma 5.1.10] (please see footnote 1) implies the desired extension, that is:

Proposition 1.3. *If R is coherent, then a finitely presented R-module is Gorenstein flat if, and only if, it is Gorenstein projective.*

[1] In [7] Christensen forgot some details in a few results (as in [7, Proposition 5.1.4]). For the corrections, see the errata on Christensen's homepage: http://www.math.unl.edu/˜1christensen3/publications.html.

In this context, the strongly Gorenstein projective and flat modules give us more relations. And we prove the two following results (see Proposition 3.9 and Corollary 3.10):

Proposition. *A module is finitely generated strongly Gorenstein projective if, and only if, it is finitely presented strongly Gorenstein flat.*

Corollary. *If R is integral domain or local, then a finitely generated R-module is strongly Gorenstein flat if, and only if, it is strongly Gorenstein projective.*

The study of finitely generated strongly Gorenstein projective and flat modules allows us to give a new characterization of S-rings.
Recall that a ring R is called an S-ring if every finitely generated flat R-module is projective (see [18]). We have Proposition 3.12:

Proposition. *R is an S-ring if, and only if, every finitely generated strongly Gorenstein flat R-module is strongly Gorenstein projective.*

Finally, to give credibility to our study, we set some examples of distinguishing the strongly Gorenstein projective, injective, and flat modules from their corresponding Gorenstein and classical modules.

2. Strongly Gorenstein projective and strongly Gorenstein injective modules

In this section we introduce and study the strongly Gorenstein projective and injective modules which are defined as follows:

Definition 2.1. A complete projective resolution of the form

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

is called strongly complete projective resolution and denoted by $(\mathbf{P}, f)$.
An R-module M is called strongly Gorenstein projective (SG-projective for short) if $M \cong \operatorname{Ker} f$ for some strongly complete projective resolution $(\mathbf{P}, f)$.
The strongly Gorenstein injective (SG-injective for short) modules are defined dually.

Using the definitions, we immediately get the following results.

Proposition 2.2. (1) *If $(P_i)_{i\in I}$ is a family of strongly Gorenstein projective modules, then $\oplus P_i$ is strongly Gorenstein projective.*
(2) *If $(I_i)_{i\in I}$ is a family of strongly Gorenstein injective modules, then $\prod I_i$ is strongly Gorenstein injective.*

Proof. Simply note that a sum (resp., product) of strongly complete projective (resp., injective) resolutions is also a strongly complete projective (resp., injective) resolution (using the natural isomorphisms in [19, Theorems 2.4 and 2.6] and [6, Section 2, N°2, Proposition 1]). ■

It is straightforward that the strongly Gorenstein projective (resp., injective) modules are a particular case of the Gorenstein projective (resp., injective) modules. And it is well known that every projective (resp., injective) module is Gorenstein projective (resp., injective). That is obtained easily by considering for a projective module P the complete projective resolution $0 \longrightarrow P \xrightarrow{=} P \longrightarrow 0$ [7, Observation 4.2.2].

The next result shows that the class of all strongly Gorenstein projective (resp., injective) modules is between the class of all projective (resp., injective) modules and the class of all Gorenstein projective (resp., injective) modules.

Proposition 2.3. *Every projective (resp., injective) module is strongly Gorenstein projective (resp., injective).*

Proof. It suffices to prove the Gorenstein projective case; the Gorenstein injective case is analogous.
Let P be a projective R-module, and consider the exact sequence

$$\begin{array}{cccccccccc} \mathbf{P} = & \cdots & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & P \oplus P & \xrightarrow{f} & \cdots \\ & & & (x, y) & \longmapsto & (0, x). & & & & \end{array}$$

We have $0 \oplus P = \operatorname{Ker} f = \operatorname{Im} f \cong P$.

Consider a projective module Q; applying the functor $\mathrm{Hom}_R(-, Q)$ to the above sequence **P**, we get the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Hom}(P \oplus P, Q) & \xrightarrow{\mathrm{Hom}_R(f,Q)} & \mathrm{Hom}(P \oplus P, Q) & \longrightarrow & \cdots \\ & & \cong\downarrow & & \cong\downarrow & & \\ \cdots & \longrightarrow & \mathrm{Hom}(P, Q) \oplus \mathrm{Hom}(P, Q) & \longrightarrow & \mathrm{Hom}(P, Q) \oplus \mathrm{Hom}(P, Q) & \longrightarrow & \cdots. \end{array}$$

Since the lower sequence in the diagram above is exact, the proposition follows. ■

The strongly Gorenstein projective (resp., injective) modules are not necessarily projective (resp., injective), as shown by the following examples. Before that, recall that a ring R is called quasi-Frobenius (a QF-ring for short) if it is Noetherian and self-injective (i.e., R is an injective R-module). For instance, if I is a nonzero ideal in a Dedekind domain R, then R/I is quasi-Frobenius [19, Exercise 9.24]. The following gives a characterization of such rings:

Theorem 2.4 (*[1], Theorem 31.9*). *The following conditions are equivalent:*

(1) *R is quasi-Frobenius;*
(2) *every projective R-module is injective;*
(3) *every injective R-module is projective.*

Now we can give the desired examples.

Example 2.5. Consider the quasi-Frobenius local ring $R = k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the residue class in R of X.

(1) The ideal $(\overline{X})$ is strongly Gorenstein projective and strongly Gorenstein injective.
(2) However it is neither projective nor injective.

Proof. (1) With the homothety x given by multiplication by $\overline{X}$ we have the exact sequence $\mathbf{F} = \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$. Then, $\mathrm{Ker}\, x = \mathrm{Im}\, x = (\overline{X})$.
Since R is quasi-Frobenius, we can see easily from Theorem 2.4 that **F** is a simultaneously strongly complete projective and injective resolution. Thus, $(\overline{X})$ is a both strongly Gorenstein projective and injective ideal.
(2) The ideal $(\overline{X})$ it is not projective, since it is not a free ideal in the local ring R (since $\overline{X}^2 = 0$). Then, from Theorem 2.4 we conclude that $\overline{X}$ is also not injective, as desired. ■

Remark 2.6. If we want to construct an example of a non-finitely generated strongly Gorenstein projective module, we can see easily, from Proposition 2.2 and using the ideal $(\overline{X})$ of the previous example, that the direct sum $(\overline{X})^{(I)}$ for any infinite index set I is a non-finitely generated strongly Gorenstein projective module.

Now we give our main result of this paper in which we give a new characterization of the Gorenstein projective (resp., injective) modules by the strongly Gorenstein projective (resp., injective) modules.

Theorem 2.7. *A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*

Proof. It suffices to prove the Gorenstein projective case; the Gorenstein injective case is analogous.
By [17, Proposition 2.5], it remains to prove the direct implication.
Let M be a Gorenstein projective module. Then, there exists a complete projective resolution

$$\mathbf{P} = \cdots \longrightarrow P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} P_{-1} \xrightarrow{d_{-1}^P} P_{-2} \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(d_0^P)$.
For all $m \in \mathbb{Z}$, denote as $\Sigma^m P$ the exact sequence obtained from **P** by increasing all indexes by m:

$$(\Sigma^m P)_i = P_{i-m} \quad \text{and} \quad d_i^{\Sigma^m P} = d_{i-m}^P \quad \text{for all } i \in \mathbb{Z}.$$

Considering the exact sequence

$$\mathbf{Q} = \oplus(\Sigma^m P) = \cdots \longrightarrow Q = \oplus P_i \xrightarrow{\oplus d_i^P} Q = \oplus P_i \xrightarrow{\oplus d_i^P} Q = \oplus P_i \longrightarrow \cdots.$$

Since $\mathrm{Im}(\oplus d_i) \cong \oplus \mathrm{Im}\, d_i$, M is a direct summand of $\mathrm{Im}(\oplus d_i)$.

Moreover, from [1, Proposition 20.2 (1)]

$$\mathrm{Hom}\left(\bigoplus_{m\in\mathbb{Z}}(\Sigma^m P), L\right) \cong \prod_{m\in\mathbb{Z}} \mathrm{Hom}(\Sigma^m P, L)$$

which is an exact sequence for any projective module L. Thus, $\mathbf{Q}$ is a strongly complete projective resolution. Therefore, M is a direct summand of the strongly Gorenstein projective module $\mathrm{Im}(\oplus d_i)$, as desired. ■

Remark 2.8. From [17, Proposition 2.4], we can consider all modules of the complete projective resolution in the previous proof to be free; and then so are the modules in the constructed strongly complete projective resolution.

At the end of this section we give an example of a Gorenstein projective module which is not strongly Gorenstein projective. Before that, we give some properties of the strongly Gorenstein projective modules.

The next result gives a simple characterization of the strongly Gorenstein projective modules.

Proposition 2.9. *For any module M, the following are equivalent:*

(1) *M is strongly Gorenstein projective;*
(2) *there exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $\mathrm{Ext}(M, Q) = 0$ for any projective module Q;*
(3) *there exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $\mathrm{Ext}(M, Q') = 0$ for any module Q' with finite projective dimension;*
(4) *there exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module; such that, for any projective module Q, the short sequence $0 \to \mathrm{Hom}(M, Q) \to \mathrm{Hom}(P, Q) \to \mathrm{Hom}(M, Q) \to 0$ is exact;*
(5) *there exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, such that, for any module Q' with finite projective dimension, the short sequence $0 \to \mathrm{Hom}(M, Q') \to \mathrm{Hom}(P, Q') \to \mathrm{Hom}(M, Q') \to 0$ is exact.*

Proof. Using standard arguments, this follows immediately from the definition of strongly Gorenstein modules. ■

Remark 2.10. (1) Note that using this characterization of strongly Gorenstein projective modules, the Proposition 2.3 becomes straightforward. Indeed, we have the short exact sequence $0 \to P \to P \oplus P \to P \to 0$, and $\mathrm{Ext}(P, Q) = 0$ for any module Q.
(2) We can also characterize the strongly Gorenstein injective modules in a way similar to the description of strongly Gorenstein projective modules in Proposition 2.9.

Recall that a strongly Gorenstein projective module is projective if, and only if, it has finite projective dimension [17, Proposition 2.27]. In the next result we give a similar result in which the strongly Gorenstein projective modules link with the flat dimension.

Corollary 2.11. *A strongly Gorenstein projective module is flat if, and only if, it has finite flat dimension.*

Proof. This is a simple consequence of Proposition 2.9. ■

The following proposition deals with finitely generated strongly Gorenstein projective modules. It is well known that a finitely generated projective module is infinitely presented (i.e., it admits a free resolution

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

such that each F_i is a finitely generated free module).
For the Gorenstein projective modules the question is still open. However, the strongly Gorenstein projective modules give the following partial affirmative answer, in which we give a characterization of the finitely generated strongly Gorenstein projective modules.

Proposition 2.12. *Let M be an R-module. The following are equivalent:*

(1) *M is finitely generated strongly Gorenstein projective;*

(2) *there exists a short exact sequence* $0 \to M \to P \to M \to 0$ *where* P *is a finitely generated projective* R*-module, and* $\mathrm{Ext}(M, R) = 0$*;*
(3) *there exists a short exact sequence* $0 \to M \to P \to M \to 0$ *where* P *is a finitely generated projective* R*-module, and* $\mathrm{Ext}(M, F) = 0$ *for all flat* R*-modules* F*;*
(4) *there exists a short exact sequence* $0 \to M \to P \to M \to 0$ *where* P *is a finitely generated projective* R*-module, and* $\mathrm{Ext}(M, F') = 0$ *for all* R*-modules* F' *with finite flat dimension.*

Proof. Note that the fourth condition is stronger than the first; this leaves us three implications to prove.
(1) $\Rightarrow$ (2). This is a simple consequence of Proposition 2.9.
(2) $\Rightarrow$ (3). Let F be a flat R-module. By Lazard's theorem [6, Section 1, N°6, Theorem 1], there is a direct system $(L_i)_{i\in I}$ of finitely generated free R-modules such that $\varinjlim L_i \cong F$. From [16, Theorem 2.1.5 (3)], we have

$$\begin{aligned}\mathrm{Ext}(M, F) &\cong \mathrm{Ext}(M, \varinjlim L_i)\\ &\cong \varinjlim \mathrm{Ext}(M, L_i).\end{aligned}$$

Now, combining [16, Theorem 2.1.5 (3)] with [19, Example 20', page 41] shows immediately that $\mathrm{Ext}(M, L_i) = 0$ for all $i \in I$, as desired.
(3) $\Rightarrow$ (4). Let F' be an R-module such that $0 < \mathrm{fd}(F') = m < \infty$.
First, we can see easily that (3) implies $\mathrm{Ext}^n(M, F) = 0$ for all $n > 0$, and all flat R-modules F.
Now, pick a short exact sequence $0 \to K \to L \to F' \to 0$ where L is a free R-module and $\mathrm{fd}(K) = m - 1$. By induction $\mathrm{Ext}^n(M, L) = \mathrm{Ext}^n(M, K) = 0$ for all $n > 0$. Then, applying the functor $\mathrm{Hom}(M, -)$ to the short exact sequence above we obtain the exact sequence

$$0 = \mathrm{Ext}(M, L) \to \mathrm{Ext}(M, F') \to \mathrm{Ext}^2(M, K) = 0.$$

Therefore, $\mathrm{Ext}(M, F') = 0$. ■

We finish this section by giving an example of a Gorenstein projective module which is not strongly Gorenstein projective.

Example 2.13. Consider the Noetherian local ring $R = k[[X_1, X_2]]/(X_1X_2)$ where k is a field. Then:

(1) The two ideals $(\overline{X_1})$ and $(\overline{X_2})$ are Gorenstein projective, where $\overline{X_i}$ is the residue class in R of X_i for $i = 1, 2$.
(2) $(\overline{X_1})$ and $(\overline{X_2})$ are not strongly Gorenstein projective.

Proof. (1) This is [7, Example 4.1.5].
(2) Assume, for example, that the ideal $(\overline{X_1})$ is strongly Gorenstein projective.

By Proposition 2.12, there exists a short exact sequence

$$0 \to (\overline{X_1}) \to P \to (\overline{X_1}) \to 0$$

where P is a finitely generated projective module. Since R is local, there exists a positive integer n such that $P \cong R^n$. Thus, we can rewrite the above short exact sequence as follows:

$$(\Phi) : 0 \longrightarrow (\overline{X_1}) \longrightarrow R^n \longrightarrow (\overline{X_1}) \longrightarrow 0.$$

On the other hand, we can see easily that we have the following short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\overline{X_2}) & \longrightarrow & R & \longrightarrow & (\overline{X_1}) & \longrightarrow & 0 \\ & & & & x & \longmapsto & x\overline{X_1}. & & \end{array}$$

Thus, by Schanuel's lemma [19, Theorem 3.62], we have $R^n \oplus (\overline{X_2}) \cong R \oplus (\overline{X_1})$.
Tensorizing by k, the residue field of R, we obtain the following isomorphism of k-vector spaces: $k^n \oplus (k \otimes_R (\overline{X_2})) \cong k \oplus (k \otimes_R (\overline{X_1}))$, and we conclude that $n = 1$. Therefore, the short exact sequence (Φ) becomes

$$0 \longrightarrow (\overline{X_1}) \xrightarrow{g} R \xrightarrow{f} (\overline{X_1}) \longrightarrow 0.$$

Now, consider $f(1) = \overline{\alpha}\overline{X_1}$ for some $\alpha \in k[[X_1, X_2]]$; hence $\operatorname{Im} f = (\overline{\alpha}\overline{X_1}) = (\overline{X_1})$, which implies that there exist β and δ in $k[[X_1, X_2]]$ such that $X_1 = \beta\alpha X_1 + \delta X_1 X_2$. Hence $\alpha\beta = 1 - \delta X_2$ which is invertible in $k[[X_1, X_2]]$; then so is α, and hence $\overline{\alpha}$ is invertible in R. Thus,

$$\operatorname{Ker} f = \{x \in R | x f(1) = x\overline{\alpha}\overline{X_1} = \overline{0}\} = \{x \in R | x\overline{X_1} = \overline{0}\} = \operatorname{Ann} \overline{X_1} = (\overline{X_2}).$$

Consequently, $(\overline{X_1}) \cong \operatorname{Im} g = \operatorname{Ker} f = (\overline{X_2})$.
But this is absurd since $\operatorname{Ann} \overline{X_1} = (\overline{X_2}) \neq (\overline{X_1}) = \operatorname{Ann} \overline{X_2}$.
Therefore, $(\overline{X_1})$ is not strongly Gorenstein projective. ■

3. Strongly Gorenstein flat modules

In this section we introduce and study the strongly Gorenstein flat modules, and further we link them with the strongly Gorenstein projective modules.

Definition 3.1. A complete flat resolution of the form

$$\mathbf{F} = \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

is called a strongly complete flat resolution and denoted by $(\mathbf{F}, f)$.
An R-module M is called strongly Gorenstein flat (SG-flat for short) if $M \cong \operatorname{Ker} f$ for some strongly complete flat resolution $(\mathbf{F}, f)$.

Consequently, the strongly Gorenstein flat modules are simple particular cases of Gorenstein flat modules. Example 3.11 gives examples of Gorenstein flat modules which are not strongly Gorenstein flat.

Now, similarly to Proposition 2.3 we prove the following:

Proposition 3.2. *Every flat module is strongly Gorenstein flat.*

Example 3.3. From Example 2.5, we can see easily that the ideal $(\overline{X})$ is also strongly Gorenstein flat, but it is not flat.

Proposition 3.4. *Every direct sum of strongly Gorenstein flat modules is also strongly Gorenstein flat.*

Proof. Immediate from the proof of Proposition 2.2 using the fact that tensor products commute with sums. ■

With strongly Gorenstein flat modules we have a simple characterization of Gorenstein flat modules, that is:

Theorem 3.5. *If a module is Gorenstein flat, then it is a direct summand of a strongly Gorenstein flat module.*

Proof. Similar to the proof of Theorem 2.7. ■

Also, similarly to Proposition 2.9, we have the following characterization of the strongly Gorenstein flat modules.

Proposition 3.6. *For any module M, the following are equivalent:*

(1) *M is strongly Gorenstein flat;*
(2) *there exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module, and $\operatorname{Tor}(M, I) = 0$ for any injective module I;*
(3) *there exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module, and $\operatorname{Tor}(M, I') = 0$ for any module I' with finite injective dimension;*
(4) *there exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module; such that the sequence $0 \to M \otimes I \to F \otimes I \to M \otimes I \to 0$ is exact for any injective module I;*
(5) *there exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is a flat module, such that the sequence $0 \to M \otimes I' \to F \otimes I' \to M \otimes I' \to 0$ is exact for any module I' with finite injective dimension.*

Holm [17, Theorem 3.19] proved, over Noetherian rings, that a Gorenstein flat module is flat if, and only if, it has a finite flat dimension. Moreover, we can see, from [17, Proposition 3.11], [16, Theorem 1.2.1], and the dual of [17, Proposition 2.27] that the same equivalence holds over coherent rings. But, in general, the question is still open. However, we can give another partial affirmative answer (Corollary 3.8). Before that, we give an affirmative answer in the case of strongly Gorenstein flat modules.

Proposition 3.7. *A strongly Gorenstein flat module is flat if, and only if, it has finite flat dimension.*

Proof. Immediate from Proposition 3.6. ■

Corollary 3.8. *If R has finite weak global dimension. Then, an R-module is Gorenstein flat if, and only if, it is flat.*

Proof. Simply combine Theorem 3.5 with Proposition 3.7. ■

From Proposition 1.3, we have that, over coherent rings, the class of all finitely presented Gorenstein projective modules and the class of all finitely presented Gorenstein flat modules are the same class. In general, the question is still open. Nevertheless, the strongly Gorenstein modules give the following partial affirmative answer:

Proposition 3.9. *A module is finitely generated strongly Gorenstein projective if, and only if, it is finitely presented strongly Gorenstein flat.*

Proof. We can prove this similarly to how [7, Lemma 5.1.10] was proved using the strongly complete resolutions (please see footnote 1). Here, we give a proof using the characterization of finitely generated strongly Gorenstein projective modules.
$\Longrightarrow$. Let M be a finitely generated strongly Gorenstein projective module. By Proposition 2.12, there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective module, and $\mathrm{Ext}(M, R) = 0$.
Let E be an injective module. Since M is infinitely presented, we have, from [16, Theorem 1.1.8], the following isomorphism:

$$\mathrm{Tor}(\mathrm{Hom}(R, E), M) \cong \mathrm{Hom}(\mathrm{Ext}(M, R), E).$$

Thus, $\mathrm{Tor}(E, M) = 0$ (since $\mathrm{Hom}(R, E) \cong E$ and $\mathrm{Ext}(M, R) = 0$). Therefore, M is strongly Gorenstein flat R-module (by Proposition 3.6).
$\Longleftarrow$. Now, assume M to be a finitely presented strongly Gorenstein flat module. From Proposition 3.6, we deduce that there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective module, and $\mathrm{Tor}(M, E) = 0$ for every injective module E. If we assume E to be faithfully injective, the same isomorphism of the direct implication above implies that $\mathrm{Ext}(M, R) = 0$. This means, by Proposition 2.12, that M is strongly Gorenstein projective. ■

It is well known that if a flat R-module M is finitely presented, or is finitely generated with R either a local or integral domain, then M is projective (see [19, Theorem 3.61 and p. 135]).
Under the same conditions we have the same relation between strongly Gorenstein flat modules and strongly Gorenstein projective modules, that is Proposition 3.9 and the following corollary:

Corollary 3.10. *If R is an integral domain or local, then a finitely generated R-module is strongly Gorenstein flat if, and only if, it is strongly Gorenstein projective.*

Proof. Use Proposition 3.9 and its proof. ■

Now, we give an example of Gorenstein flat modules which are not strongly Gorenstein flat.

Example 3.11. Consider the Noetherian local ring $R = k[[X_1, X_2]]/(X_1X_2)$ where k is a field. Then, the two ideals $(\overline{X_1})$ and $(\overline{X_2})$ are Gorenstein flat, where $\overline{X_i}$ is the residue class in R of X_i for $i = 1, 2$. But, they are not strongly Gorenstein flat.

Proof. Simply apply [7, Theorem 5.1.11] and Proposition 3.9 to Example 2.13. ■

In studying perfect rings, Bass [5] proved that a ring R is perfect if, and only if, every flat R-module is projective (see also [1,22] for more details about this ring).
Motivated by this result, Sakhajev asked when, more generally, every finitely generated flat module is projective (see [20]). In fact, the early study of this question goes back to the 60s, namely with the considerable works of Vasconcelos [21] and Endo [9]. However, a first general answer appeared with Facchini et al. [14]. Recently, an excessive study of it was made by Puninski and Rothmaler [18], who called the ring which satisfies the question an S-ring, to honor Sakhajev.

Now it is natural to ask: When is every finitely generated strongly Gorenstein flat module strongly Gorenstein projective?

The answer of this question gives a new characterization of S-rings, that is:

Proposition 3.12. *R is an S-ring if, and only if, every finitely generated strongly Gorenstein flat R-module is strongly Gorenstein projective.*

Proof. $\Longrightarrow$. Let M be a finitely generated strongly Gorenstein flat R-module. Then, by Proposition 3.6, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where F is a finitely generated flat R-module. By hypothesis F is projective, and so M is finitely presented. Therefore, from Proposition 3.9, M is strongly Gorenstein projective.

$\Longrightarrow$. Now, assume M to be a finitely generated flat R-module. Then, from Proposition 3.2, M is finitely generated strongly Gorenstein flat. Hence, it is, by hypothesis, strongly Gorenstein projective. Thus, from Proposition 2.12, there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a finitely generated projective R-module, and $\mathrm{Ext}(M, F) = 0$ for all flat R-modules F. Then, $\mathrm{Ext}(M, M) = 0$ (since M is flat), and then the above short exact sequence splits. Therefore, M is projective as a direct summand of the projective R-module P, as desired. ■

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References

[1] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, 2nd edition, in: Graduate Texts Mathematics, vol. 13, Springer-Verlag, New York, 1992.
[2] M. Auslander, Anneaux de Gorenstein et torsion en algèbre commutative, Secrétariat mathématique, Paris, 1967, in: Séminaire d'algèbre commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro, Ecole Normale Superieure de Jeunes Filles.
[3] M. Auslander, M. Bridger, Stable module theory, in: Memoirs. Amer. Math. Soc., vol. 94, American Mathematical Society, Providence, RI, 1969.
[4] L.L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. (3) 85 (2) (2002) 393–440.
[5] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960) 466–488.
[6] N. Bourbaki, Algèbre Homologique, Masson, Paris, 1980 (Chapitre 10).
[7] L.W. Christensen, Gorenstein Dimensions, in: Lecture Notes in Math., vol. 1747, Springer, Berlin, 2000.
[8] L.W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions — a functorial description with applications, J. Algebra 302 (2006) 231–279.
[9] Sh. Endo, On flat modules over commutative rings, J. Math. Soc. Japan 14 (3) (1962) 284–291.
[10] E. Enochs, O. Jenda, On Gorenstein injective modules, Comm. Algebra 21 (10) (1993) 3489–3501.
[11] E. Enochs, O. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (4) (1995) 611–633.
[12] E. Enochs, O. Jenda, B. Torrecillas, Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan 10 (1) (1993) 1–9.
[13] E. Enochs, O. Jenda, J. Xu, Foxby duality and Gorenstein injective and projective modules, Trans. Amer. Math. Soc. 348 (8) (1996) 3223–3234.
[14] A. Facchini, D. Herbera, I. Sakhajev, Finitely generated modules and a characterization of semiperfect rings, Comm. Algebra 31 (2003) 4195–4215.
[15] H.-B. Foxby, Gorenstein dimension over Cohen–Macaulay rings, in: W. Bruns (Ed.), Proceedings of International Conference on Commutative Algebra, Universität Onsabrück, 1994.
[16] S. Glaz, Commutative Coherent Rings, in: Lecture Notes in Mathematics, vol. 1371, Springer-Verlag, 1989.
[17] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (2004) 167–193.
[18] G. Puninski, P. Rothmaler, When every finitely generated flat module is projective, J. Algebra 277 (2004) 542–558.
[19] J. Rotman, An Introduction to Homological Algebra, in: Pure and Appl. Math., A Series of Monographs and Textbooks, vol. 25, Academic Press, 1979.
[20] I. Sakhajev, On a criterion of projectivity of finitely generated flat modules, Izv. Vuzov 10 (1991) 68–75.
[21] W.V. Vasconcelos, On finitely generated flat modules, Trans. Amer. Math. Soc. 138 (1969) 505–512.
[22] J. Xu, Flat Covers of Modules, in: Lecture Notes in Mathematics, vol. 1634, Springer-Verlag, 1996.